

CHARACTER SHEAVES AND GENERALIZATIONS

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Dedicated to I. M. Gelfand on the occasion of his 90th birthday

1. Let \mathbf{k} be an algebraic closure of a finite field \mathbf{F}_q . Let $G = GL_n(\mathbf{k})$. The group $G(\mathbf{F}_q) = GL_n(\mathbf{F}_q)$ can be regarded as the fixed point set of the Frobenius map $F : G \rightarrow G, (g_{ij}) \mapsto (g_{ij}^q)$. Let $\bar{\mathbf{Q}}_l$ be an algebraic closure of the field of l -adic numbers, where l is a prime number invertible in \mathbf{k} . The characters of irreducible representations of $G(\mathbf{F}_q)$ over an algebraically closed field of characteristic 0, which we take to be $\bar{\mathbf{Q}}_l$, have been determined explicitly by J.A.Green [G]. The theory of character sheaves [L2] tries to produce some geometric objects over G from which the irreducible characters of $G(\mathbf{F}_q)$ can be deduced for any q . This allows us to unify the representation theories of $G(\mathbf{F}_q)$ for various q . The geometric objects needed in the theory are provided by intersection cohomology.

Let X be an algebraic variety over \mathbf{k} , let X_0 be a locally closed irreducible, smooth subvariety of X and let \mathcal{E} be a local system over X_0 (we say "local system" instead of " $\bar{\mathbf{Q}}_l$ -local system"). Deligne, Goresky and MacPherson attach to this datum a canonical object $IC(\bar{X}_0, \mathcal{E})$ (intersection cohomology complex) in the derived category $\mathcal{D}(X)$ of $\bar{\mathbf{Q}}_l$ -sheaves on X ; this is a complex of sheaves which extends \mathcal{E} to X (by 0 outside the closure \bar{X}_0 of X_0) in the most economical possible way so that local Poicaré duality is satisfied. We say that $IC(\bar{X}_0, \mathcal{E})$ is irreducible if \mathcal{E} is irreducible.

Now take $X = G$ and take $X_0 = G_{rs}$ to be the set of regular semisimple elements in G . Let T be the group of diagonal matrices in G . For any integer $m \geq 1$ invertible in \mathbf{k} we have an unramified $n!m^n$ -fold covering

$$\pi_m : \{(g, t, xT) \in G_{rs} \times T \times G/T; x^{-1}gx = t^m\} \rightarrow G_{rs}, \quad (g, t, xT) \mapsto g.$$

An irreducible local system \mathcal{E} on G_{rs} is said to be admissible if it is a direct summand of the local system $\pi_{m!}\bar{\mathbf{Q}}_l$ for some m as above. The character sheaves on G are the complexes $IC(G, \mathcal{E})$ for various admissible local systems \mathcal{E} on G_{rs} .

We show how the irreducible characters of $G(\mathbf{F}_q)$ can be recovered from character sheaves on G . If A is a character sheaf on G then its inverse image F^*A under F is again a character sheaf. There are only finitely many A (up to isomorphism) such that F^*A is isomorphic to A . For any such A we choose an isomorphism

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$\phi : F^*A \xrightarrow{\sim} A$ and we form the characteristic function $\chi_{A,\phi} : G(\mathbf{F}_q) \rightarrow \bar{\mathbf{Q}}_l$ whose value at g is the alternating sum of traces of ϕ on the stalks at g of the cohomology sheaves of A . Now ϕ is unique up to a non-zero scalar hence $\chi_{A,\phi}$ is unique up to a non-zero scalar. It turns out that

(a) $\chi_{A,\phi}$ is (up to a non-zero scalar) the character of an irreducible representation of $G(\mathbf{F}_q)$ and $A \mapsto \chi_{A,\phi}$ gives a bijection between the set of (isomorphism classes of) character sheaves on G that are isomorphic to their inverse image under F and the irreducible characters of $G(\mathbf{F}_q)$.

(This result is essentially contained in [L1,L3].) The main content of this result is that the (rather complicated) values of an irreducible character of $G(\mathbf{F}_q)$ are governed by a geometric principle, namely by the procedure which gives the intersection cohomology extension of a local system.

2. More generally, assume that G is a connected reductive algebraic group over \mathbf{k} . The definition of the $IC(G, \mathcal{E})$ given above for GL_n makes sense also in the general case. The complexes on G obtained in this way form the class of *uniform* character sheaves on G . Consider now a fixed \mathbf{F}_q -rational structure on G with Frobenius map $F : G \rightarrow G$. The analogue of property 1(a) does not hold in general for (G, F) . It is still true that the characteristic functions of the uniform character sheaves that are isomorphic to their inverse image under F are linearly independent class functions $G(\mathbf{F}_q) \rightarrow \bar{\mathbf{Q}}_l$. However they do not form a basis of the space of class functions. Moreover they are in general not irreducible characters of $G(\mathbf{F}_q)$ (up to a scalar); rather, each of them is a linear combination with known coefficients of a "small" number of irreducible characters of $G(\mathbf{F}_q)$ (where "small" means "bounded independently of q "); this result is essentially contained in [L1,L3].

It turns out that the class of uniform character sheaves can be naturally enlarged to a larger class of complexes on G .

For any parabolic P of G , U_P denotes the unipotent radical of P . For a Borel B in G , the images under $c^B : G \rightarrow G/U_B$ of the double cosets BwB form a partition $G/U_B = \cup_w (BwB/U_B)$.

An irreducible intersection cohomology complex $A \in \mathcal{D}(G)$ is said to be a character sheaf on G if it is G -equivariant and if for some/any Borel B in G , $c_!^B A$ has the following property:

(*) *any cohomology sheaf of this complex restricted to any BwB/U_B is a local system with finite monodromy of order invertible in \mathbf{k} .*

Then any uniform character sheaf on G is a character sheaf on G . For $G = GL_n$ the converse is also true, but for general G this is not so.

Consider again a fixed \mathbf{F}_q -rational structure on G with Frobenius map $F : G \rightarrow G$. The following partial analogue of property 1(a) holds (under a mild restriction on the characteristic of \mathbf{k}).

(a) *The characteristic functions of the various character sheaves A on G (up to isomorphism) such that $F^*A \xrightarrow{\sim} A$ form a basis of the vector space of class functions $G(\mathbf{F}_q) \rightarrow \bar{\mathbf{Q}}_l$.*

3. We now fix a parabolic P of G . For any Borel B of P let $\tilde{c}^B : G/U_P \rightarrow G/U_B$ be the obvious map. Now P acts on G/U_P by conjugation.

An irreducible intersection cohomology complex $A \in \mathcal{D}(G/U_P)$ is said to be a parabolic character sheaf if it is P -equivariant and if for some/any Borel B in P , $\tilde{c}_!^B A$ has property 2(*). When $P = G$, we recover the definition of character sheaves on G .

Consider now a fixed \mathbf{F}_q -rational structure on G with Frobenius map $F : G \rightarrow G$ such that P is defined over \mathbf{F}_q . Then G/U_P has a natural \mathbf{F}_q -rational structure with Frobenius map F . The following generalization of 2(a) holds (under a mild restriction on the characteristic of \mathbf{k}).

(a) *The characteristic functions of the various parabolic character sheaves A on G/U_P (up to isomorphism) such that $F^* A \xrightarrow{\sim} A$ form a basis of the vector space \mathcal{V} of $P(\mathbf{F}_q)$ -invariant functions $G(\mathbf{F}_q)/U_P(\mathbf{F}_q) \rightarrow \bar{\mathbf{Q}}_l$.*

The proof is given in [L5]. It relies on a generalization of property 2(a) to not necessarily connected reductive groups which will be contained in the series [L6].

If $h : G(\mathbf{F}_q) \rightarrow \bar{\mathbf{Q}}_l$ is the characteristic function of a character sheaf as in 2(a) then by summing h over the fibres of $G(\mathbf{F}_q) \rightarrow G(\mathbf{F}_q)/U_P(\mathbf{F}_q)$ we obtain a function $\bar{h} \in \mathcal{V}$. It turns out that each function \bar{h} is a linear combination of a "small" number of elements in the basis of \mathcal{V} described above. (The fact such a basis of \mathcal{V} exists is not apriori obvious.)

The parabolic character sheaves on G/U_P are expected to be a necessary ingredient in establishing the conjectural geometric interpretation of Hecke algebras with unequal parameters given in [L4].

4. In this section G denotes an abelian group with a given family \mathfrak{F} of automorphisms such that

- (i) if $F \in \mathfrak{F}$ and $n \in \mathbf{Z}_{>0}$, then $F^n \in \mathfrak{F}$;
- (ii) if $F \in \mathfrak{F}, F' \in \mathfrak{F}$ then there exist $n, n' \in \mathbf{Z}_{>0}$ such that $F^n = F'^{n'}$;
- (iii) for any $F \in \mathfrak{F}$, the map $G \rightarrow G, x \mapsto F(x)x^{-1}$ is surjective with finite kernel.

For $F \in \mathfrak{F}$ and $n \in \mathbf{Z}_{>0}$, the homomorphism

$$N_{F^n/F} : G \rightarrow G, x \mapsto xF(x) \dots F^{n-1}(x),$$

restricts to a surjective homomorphism $G^{F^n} \rightarrow G^F$. (If $y \in G^F$ we can find $z \in G$ with $y = F^n(z)z^{-1}$, by (i),(iii). We set $x = F(z)z^{-1}$. Then $x \in G^{F^n}$ and $N_{F^n/F}(x) = y$.) Let X be the set of pairs (F, ψ) where $F \in \mathfrak{F}$ and $\psi \in \text{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$. Consider the equivalence relation on X generated by $(F, \psi) \sim (F^n, \psi \circ N_{F^n/F})$. Let G^* be the set of equivalence classes. We define a group structure on G^* . We consider two elements of G^* ; we represent them in the form $(F, \psi), (F', \psi')$ where $F = F'$ (using (ii)) and we define their product as the equivalence class of $(F, \psi\psi')$; one checks that this product is independent of the choices. This makes G^* into an abelian group. The unit element is the equivalence class of $(F, 1)$ for any $F \in \mathfrak{F}$. For $F \in \mathfrak{F}$ we define an automorphism $F^* : G^* \rightarrow G^*$ by sending an element of G^* represented by (F^n, ψ) with $n \in$

$\mathbf{Z}_{>0}$, $\psi \in \text{Hom}(G^{F^n}, \bar{\mathbf{Q}}_l^*)$ to $(F^n, \psi \circ F)$ (here $\psi \circ F$ is the composition $G^{F^n} \xrightarrow{F} G^{F^n} \xrightarrow{\psi} \bar{\mathbf{Q}}_l^*$); one checks that this is well defined. For any $F \in \mathfrak{F}$ the map $\text{Hom}(G^F, \bar{\mathbf{Q}}_l^*) \rightarrow G^*$, $\psi \mapsto (F, \psi)$ is

(a) *a group isomorphism of $\text{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ onto the subgroup $(G^*)^{F^*}$ of G^* .*
 (This follows from the surjectivity of $N_{F^n/F} : G^{F^n} \rightarrow G^F$.)

5. Assume now that G is an abelian, connected (affine) algebraic group over \mathbf{k} . We define the notion of character sheaf on G .

Let \mathfrak{F} be the set of Frobenius maps $F : G \rightarrow G$ for various rational structures on G over a finite subfield of \mathbf{k} . (These maps are automorphisms of G as an abstract group.) Then properties 4(i)-4(iii) are satisfied for (G, \mathfrak{F}) hence the abelian group G^* is defined as in §4. We will give an interpretation of G^* in terms of local systems on G . Let $F \in \mathfrak{F}$. Let $L : G \rightarrow G$ be the Lang map $x \mapsto F(x)x^{-1}$. Consider the local system $E = L^* \bar{\mathbf{Q}}_l$ on G . Its stalk at $y \in G$ is the vector space E_y consisting of all functions $f : L^{-1}(y) \rightarrow \bar{\mathbf{Q}}_l$. We have $E_y = \bigoplus_{\psi \in \text{Hom}(G^F, \bar{\mathbf{Q}}_l^*)} E_y^\psi$ where

$$E_y^\psi = \{f \in E_y; f(zx) = \psi(z)f(x) \quad \forall z \in G^F, x \in L^{-1}(y)\}.$$

We have a canonical direct sum decomposition $E = \bigoplus_{\psi} E^\psi$ where E^ψ is a local system of rank 1 on G whose stalk at $y \in G$ is E_y^ψ (ψ as above). There is a unique isomorphism of local systems $\phi : F^* E^\psi \xrightarrow{\sim} E^\psi$ which induces identity on the stalk at 1. This induces for any $y \in G$ the isomorphism $E_{F(y)}^\psi \rightarrow E_y^\psi$ given by $f \mapsto f'$ where $f'(x) = f(F(x))$. If $y \in G^F$, this isomorphism is multiplication by $\psi(y)$. Thus, the characteristic function $\chi_{E^\psi, \phi} : G^F \rightarrow \bar{\mathbf{Q}}_l$ is the character ψ .

Let $n \in \mathbf{Z}_{>0}$. Let $L' : G \rightarrow G$ be the map $x \mapsto F^n(x)x^{-1}$. Consider the local system $E' = L'^* \bar{\mathbf{Q}}_l$ on G . Its stalk at $y \in G$ is the vector space E'_y consisting of all functions $f' : L'^{-1}(y) \rightarrow \bar{\mathbf{Q}}_l$. We define $E_y \rightarrow E'_y$ by $f \mapsto f'$ where $f'(x) = f(N_{F^n, F} x)$ (note that $N_{F^n, F}(L'^{-1}(y)) \subset L^{-1}(y)$). This is induced by a morphism of local systems $E \rightarrow E'$ which restricts to an isomorphism $E^\psi \xrightarrow{\sim} E'^{\psi'}$ where $\psi' = \psi \circ N_{F^n, F} \in \text{Hom}(G^{F^n}, \bar{\mathbf{Q}}_l^*)$.

From the definitions we see that, if $\psi, \psi' \in \text{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ then for any $y \in G$ we have an isomorphism $E_y^\psi \otimes E_y^{\psi'} \xrightarrow{\sim} E_y^{\psi\psi'}$ given by multiplication of functions on $L^{-1}(y)$. This comes from an isomorphism of local systems $E^\psi \otimes E^{\psi'} \xrightarrow{\sim} E^{\psi\psi'}$.

A *character sheaf* on G is by definition a local system of rank 1 on G of the form E^ψ for some (F, ψ) as above. Let $\mathcal{S}(G)$ be the set of isomorphism classes of character sheaves on G . Then $\mathcal{S}(G)$ is an abelian group under tensor product. The arguments above show that $(F, \psi) \mapsto E^\psi$ defines a (surjective) group homomorphism $G^* \rightarrow \mathcal{S}(G)$. This is in fact an isomorphism. (It is enough to show that, if (F, ψ) is as above and $\psi' \in \text{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ is such that the local systems $E^\psi, E^{\psi'}$ are isomorphic, then $\psi = \psi'$. As we have seen earlier, each of $E^\psi, E^{\psi'}$ has a unique isomorphism ϕ, ϕ' with its inverse image under $F : G \rightarrow G$ which induces the identity at the stalk at 1. Then we must have $\chi_{E^\psi, \phi} = \chi_{E^{\psi'}, \phi'}$ hence $\psi = \psi'$. Note that for $F \in \mathfrak{F}$, the map $F^* : G^* \rightarrow G^*$ corresponds under the isomorphism

$G^* \xrightarrow{\sim} \mathcal{S}(G)$ to the map $\mathcal{S}(G) \rightarrow \mathcal{S}(G)$ given by inverse image under F . Using this and 4(a), we see that, for $F \in \mathfrak{F}$, the map $\text{Hom}(G^F, \bar{\mathbf{Q}}_l^*) \rightarrow \mathcal{S}(G), \psi \mapsto E^\psi$ is a group isomorphism of $\text{Hom}(G^F, \bar{\mathbf{Q}}_l^*)$ onto the subgroup of $\mathcal{S}(G)$ consisting of all character sheaves on G that are isomorphic to their inverse image under F . We see that in this case the analogue of 1(a) holds.

From the definitions, we see that,

(a) if $\mathcal{L}_1 \in \mathcal{S}(G)$ and $m : G \times G \rightarrow G$ is the multiplication map then $m^* \mathcal{L}_1 = \mathcal{L}_1 \otimes \mathcal{L}_1$.

In the case where $G = \mathbf{k}$, our definition of character sheaves on G reduces to that of the Artin-Schreier local systems on \mathbf{k} .

6. In this section we assume that G is a unipotent algebraic group over \mathbf{k} of "exponential type" that is, such that the exponential map from $\text{Lie } G$ to G is well defined (and an isomorphism of varieties.) In this case we can define character sheaves on G using Kirillov theory. Namely, for each G -orbit in the dual of $\text{Lie } G$ we consider the local system $\bar{\mathbf{Q}}_l$ on that orbit extended by 0 on the complement of the orbit. Taking the Fourier-Deligne transform we obtain (up to shift) an irreducible intersection cohomology complex on $\text{Lie } G$ (since the orbit is smooth and closed, by Kostant-Rosenlicht). We can view it as an intersection cohomology complex on G via the exponential map. The complexes on G thus obtained are by definition the character sheaves of G . Using Kirillov theory (see [K]) we see that in this case the analogue of 1(a) holds.

Assume, for example, that G is the group of all matrices

$$[a, b, c] = \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

with entries in \mathbf{k} and that $2^{-1} \in \mathbf{k}$. Consider the following intersection cohomology complexes on G :

(i) the complex which on the centre $\{(0, b, 0); b \in \mathbf{k}\}$ is the local system $\mathcal{E} \in \mathcal{S}(\mathbf{k}), \mathcal{E} \neq \bar{\mathbf{Q}}_l$ wxtended by 0 to the whole of G ;

(ii) the local system $f^* \mathcal{E}$ where $f[a, b, c] = (a, c)$ and $\mathcal{E} \in \mathcal{S}(\mathbf{k}^2)$.

The complexes (i),(ii) are the character sheaves of G .

7. In this section we assume that G is a connected unipotent algebraic group over \mathbf{k} (not necessarily of exponential type). We expect that in this case there is again a notion of character sheaf on G such that over a finite field, the characteristic functions of character sheaves form a basis of the space of class functions and each characteristic function of a character sheaf is a linear combination of a "small" number of irreducible characters. Thus here the situation should be similar to that for a general connected reductive group rather than that for GL_n . We illustrate this in one example. Assume that \mathbf{k} has characteristic 2. Let G be the group

consisting of all matrices of the form

$$\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & b+ad \\ 0 & 0 & 1 & a \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

with entries in \mathbf{k} ; we also write $[a, b, c, d]$ instead of the matrix above. (This group can be regarded as the unipotent radical of a Borel in $Sp_4(\mathbf{k})$.)

Let $\mathcal{E}_0 \in \mathcal{S}(\mathbf{k})$ be the local system on \mathbf{k} associated in §5 to \mathbf{F}_q and to the homomorphism $\psi_0 : \mathbf{F}_q \rightarrow \bar{\mathbf{Q}}_l^*$ (composition of the trace $\mathbf{F}_q \rightarrow \mathbf{F}_2$ and the unique injective homomorphism $\mathbf{F}_2 \rightarrow \bar{\mathbf{Q}}_l^*$).

Consider the following intersection cohomology complexes on G :

(i) the complex which on the centre $\{[0, b, c, 0]; (b, c) \in \mathbf{k}^2\}$ is the local system $\mathcal{E} \in \mathcal{S}(\mathbf{k}^2), \mathcal{E} \neq \bar{\mathbf{Q}}_l$ (see §5) extended by 0 to the whole of G ;

(ii) the complex which on $\{[a_0, b, c, 0]; (b, c) \in \mathbf{k}^2\}$ (with $a_0 \in \mathbf{k}^*$ fixed) is the local system $pr_c^* \mathcal{E}$ where $\mathcal{E} \in \mathcal{S}(\mathbf{k}), \mathcal{E} \neq \bar{\mathbf{Q}}_l$ (see §5) extended by 0 to the whole of G ;

(iii) the complex which on $\{[0, b, c, d_0]; (b, c) \in \mathbf{k}^2\}$ (with $d_0 \in \mathbf{k}^*$ fixed) is the local system $f^* \mathcal{E}_0$ where $f[0, b, c, d_0] = \alpha b + \alpha^2 d_0 c$ (with $\alpha \in \mathbf{k}^*$ fixed) extended by 0 to the whole of G ;

(iv) the complex which on $\{[a_0, b, c, d_0]; (b, c) \in \mathbf{k}^2\}$ (with $a_0, d_0 \in \mathbf{k}^*$ fixed) is the local system $f^* \mathcal{E}_0$ where $f[a_0, b, c, d_0] = a_0^{-2} d_0^{-1} c$ extended by 0 to the whole of G ;

(v) the local system $f^* \mathcal{E}$ on G where $f[a, b, c, d] = (a, d) \in \mathbf{k}^2$ and $\mathcal{E} \in \mathcal{S}(\mathbf{k}^2)$.

By definition, the character sheaves on G are the complexes in (i)-(v) above. Note that there are infinitely many subvarieties of G which appear as supports of character sheaves (this in contrast with the case of reductive groups). There is a symmetry that exchanges the character sheaves of type (ii) with those of type (iii). Namely, define $\xi : G \rightarrow G$ by

$$[a, b, c, d] \mapsto [d, c + ab + a^2 d, b^2 + dc + abd, a^2].$$

Then ξ is a homomorphism whose square is $[a, b, c, d] \mapsto [a^2, b^2, c^2, d^2]$; moreover, ξ^* interchanges the sets (ii) and (iii) and it leaves stable each of the sets (i), (iv) and (v).

Now G has an obvious \mathbf{F}_q -structure with Frobenius map $F : G \rightarrow G$. We describe the irreducible characters of $G(\mathbf{F}_q)$.

(I) We have q^2 one dimensional characters $U \rightarrow \bar{\mathbf{Q}}_l^*$ of the form $[a, b, c, d] \mapsto \psi_0(xa + yd)$ (one for each $x, y \in \mathbf{F}_q$).

(II) We have $q - 1$ irreducible characters of degree q of the form $[0, b, c, 0] \mapsto q\psi_0(xb)$ (all other elements are mapped to 0), one for each $x \in \mathbf{F}_q - \{0\}$.

(III) We have $q - 1$ irreducible characters of degree q of the form $[0, b, c, 0] \mapsto q\psi_0(xc)$ (all other elements are mapped to 0), one for each $x \in \mathbf{F}_q - \{0\}$.

(IV) We have $4(q-1)^2$ irreducible characters of degree $q/2$, one for each quadruple $(a_0, d_0, \epsilon_1, \epsilon_2)$ where

$a_0 \in \mathbf{F}_q^*, d_0 \in \mathbf{F}_q^*, \epsilon_1 \in \text{Hom}(\{0, a_0\}, \pm 1), \epsilon_2 \in \text{Hom}(\{0, d_0\}, \pm 1)$, namely

$$[a, b, c, d] \mapsto (q/2)\epsilon_1(a)\epsilon_2(d)\psi_0(a_0^{-2}d_0^{-1}(ba + ba_0 + c)),$$

if $a \in \{0, a_0\}, d \in \{0, d_0\}$; all other elements are sent to 0.

A character of type (II) is obtained by inducing from the subgroup $\{[a, b, c, d] \in G(\mathbf{F}_q); d = 0\}$ the one dimensional character $[a, b, c, 0] \mapsto \psi_0(xb)$ where $x \in \mathbf{F}_q - \{0\}$. A character of type (III) is obtained by inducing from the commutative subgroup $\{[a, b, c, d] \in G(\mathbf{F}_q); a = 0\}$ the one dimensional character $[0, b, c, d] \mapsto \psi_0(xc)$ where $x \in \mathbf{F}_q - \{0\}$. A character of type (IV) is obtained by inducing from the subgroup $\{(a, b, c, d) \in G(\mathbf{F}_q); a \in \{0, a_0\}\}$ (where $a_0 \in \mathbf{F}_q - \{0\}$ is fixed) the one dimensional character $[a, b, c, d] \mapsto \epsilon_1(a)\psi_0(fd + a_0^{-2}d_0^{-1}(ba + ba_0 + c))$ where $f \in \mathbf{F}_q$ is chosen so that $\psi_0(fd_0) = \epsilon_2(d_0)$ (the induced character does not depend on the choice of f).

Consider the matrix expressing the characteristic functions of character sheaves A such that $F^*A \cong A$ (suitably normalized) in terms of irreducible characters of $G(\mathbf{F}_q)$. This matrix is square and a direct sum of diagonal blocks of size 1×1 (with entry 1) or 4×4 with entries $\pm 1/2$, representing the Fourier transform over a two dimensional symplectic \mathbf{F}_2 -vector space. There are $(q-1)^2$ blocks of size 4×4 involving the irreducible characters of type IV.

We see that, in our case, the character sheaves have the desired properties. We also note that in our case, $G(\mathbf{F}_q)$ has some irreducible character whose degree is not a power of q (but $q/2$) in contrast with what happens in the situation in §6.

8. Let ϵ be an indeterminate. For $r \geq 2$ let $\mathcal{A}_r = \mathbf{k}[\epsilon]/(\epsilon^r)$. Let $G = GL_n(\mathcal{A}_r)$. Let B (resp. T) be the group of upper triangular (resp. diagonal) matrices in G . Then G is in a natural way a connected affine algebraic group over \mathbf{k} of dimension n^2r and B, T are closed subgroups of G . On G we have a natural \mathbf{F}_q -structure with Frobenius map $F : G \rightarrow G, (g_{ij}) \mapsto (g_{ij}^{(q)})$ where for a_0, a_1, \dots, a_{r-1} in \mathbf{k} we set $(a_0 + a_1\epsilon + \dots + a_{r-1}\epsilon^{r-1})^{(q)} = a_0^q + a_1^q\epsilon + \dots + a_{r-1}^q\epsilon^{r-1}$. The fixed point set of $F : G \rightarrow G$ is $GL_n(\mathbf{F}_q[\epsilon]/(\epsilon^r))$. For $i \neq j$ in $[1, n]$, we consider the homomorphism $f_{ij} : \mathbf{k} \rightarrow T$ which takes $x \in \mathbf{k}$ to the diagonal matrix with ii -entry equal to $1 + \epsilon^{r-1}x$, jj -entry equal to $1 - \epsilon^{r-1}x$ and other diagonal entries equal to 1. Since T is connected and commutative, the group $\mathcal{S}(T)$ is defined (see §5). Let $\mathcal{L} \in \mathcal{S}(T)$. We will assume that \mathcal{L} is *regular* in the following sense: for any $i \neq j$ in $[1, n]$, $f_{ij}^*\mathcal{L}$ is not isomorphic to \mathbf{Q}_l .

Let $\pi : B \rightarrow T$ be the obvious homomorphism. Consider the diagram

$$G \xleftarrow{a} Y \xrightarrow{b} T$$

where

$$Y = \{(g, xB) \in G \times G/B; x^{-1}gx \in B\}, a(g, xB) = g, b(g, xB) = \pi(x^{-1}gx).$$

Then $b^*\mathcal{L}$ is a local system on Y and we may consider the complex $a_!b^*\mathcal{L}$ on G .

As in §5, we can find an integer $m_0 > 0$ such that, for any $m \in \mathcal{M} = \{m_0, 2m_0, 3m_0, \dots\}$, \mathcal{L} is associated to $(\mathbf{F}_{q^m}, \psi_m)$ where $\psi_m \in \text{Hom}(T^{F^m}, \bar{\mathbf{Q}}_l^*)$. We can regard ψ_m as a character $B(\mathbf{F}_{q^m}) \rightarrow \bar{\mathbf{Q}}_l^*$ via $\pi : B \rightarrow T$; inducing this from $B(\mathbf{F}_{q^m})$ to $G(\mathbf{F}_{q^m})$ we obtain a representation of $G(\mathbf{F}_{q^m})$ whose character is denoted by c_m . It is easy to see (using the regularity of \mathcal{L}) that this character is irreducible.

For $m \in \mathcal{M}$, there is a unique isomorphism $(F^m)^*\mathcal{L} \xrightarrow{\sim} \mathcal{L}$ of local systems on T which induces the identity on the stalk of \mathcal{L} at 1. This induces an isomorphism $(F^m)^*(b^*\mathcal{L}) \xrightarrow{\sim} b^*\mathcal{L}$ (where $F : Y \rightarrow Y$ is $(g, xB) \mapsto (F(g), F(x)B)$) and an isomorphism $(F^m)^*(a_!b^*\mathcal{L}) \xrightarrow{\sim} a_!b^*\mathcal{L}$ in $\mathcal{D}(G)$. Let $\chi_m : G^{F^m} \rightarrow \bar{\mathbf{Q}}_l^*$ be the characteristic function of $a_!b^*\mathcal{L}$ with respect to this isomorphism. From the definitions we see that $\chi_m = c_m$. This shows that $a_!b^*\mathcal{L}$ behaves like a character sheaf except for the fact that it is not clear that it is an intersection cohomology complex.

We conjecture that:

(a) *if \mathcal{L} is regular then $a_!b^*\mathcal{L}$ is an intersection cohomology complex on G .*

(The conjecture also makes sense and is expected to be true when GL_n is replaced by any reductive group, and G by the corresponding group over \mathcal{A}_r .) Thus one can expect that there is a theory of character sheaves for G , as far as generic principal series representations and their twisted forms is concerned. But one cannot expect a complete theory of character sheaves in this case (see §13).

In §9-§12 we prove the conjecture in the special case where $G = GL_2(\mathbf{k})$ and $r = 2$.

9. Let $\mathcal{A} = \mathcal{A}_2 = \mathbf{k}[\epsilon]/(\epsilon^2)$. Let V be a free \mathcal{A} -module of rank 2. Let G be the group of automorphisms of the \mathcal{A} -module V . This is the group of all automorphisms of the 4-dimensional \mathbf{k} -vector space V that commute with the map $\epsilon : V \rightarrow V$ given by the \mathcal{A} -module structure. Hence G is an algebraic group of dimension 8 over \mathbf{k} . Let ${}^0\tilde{G}$ be the set of all pairs (g, V_2) where $g \in G$ and V_2 is a free \mathcal{A} -submodule of V of rank 1 such that $gV_2 = V_2$. For $k = 1, 2$, let X_k be the set of all \mathcal{A} -submodules of V that have dimension k as a \mathbf{k} -vector space. Let \tilde{G} be the set of all triples (g, V_1, V_2) where $g \in G$, $V_1 \in X_1$, $V_2 \in X_2$, $V_1 \subset V_2$, $gV_1 = V_1$, $gV_2 = V_2$ and the scalars by which g acts on V_1 and V_2/V_1 coincide. We can regard ${}^0\tilde{G}$ as a subset of \tilde{G} by $(g, V_2) \mapsto (g, \epsilon V_2, V_2)$. Note that \tilde{G} is naturally an algebraic variety over \mathbf{k} and ${}^0\tilde{G}$ is an open subset of \tilde{G} .

The group of units \mathcal{A}' of \mathcal{A} is an algebraic group isomorphic to $\mathbf{k}^* \times \mathbf{k}$. Hence $\mathcal{S}(\mathcal{A}')$ is defined. Let $\mathcal{L}_1 \in \mathcal{S}(\mathcal{A}')$, $\mathcal{L}_2 \in \mathcal{S}(\mathcal{A}')$. Let $\mathcal{L} = \mathcal{L}_1 \boxtimes \mathcal{L}_2 \in \mathcal{S}(\mathcal{A}' \times \mathcal{A}')$, $\mathcal{E} = \mathcal{L}_2 \otimes \mathcal{L}_1^* \in \mathcal{S}(\mathcal{A}')$. Define $f : {}^0\tilde{G} \rightarrow \mathcal{A}' \times \mathcal{A}'$ by $f(g, V_2) = (\alpha_1, \alpha_2)$ where $\alpha_1 \in \mathcal{A}'$ is given by $gv = \alpha_1 v$ for $v \in V_2$ and $\alpha_2 \in \mathcal{A}'$ is given by $gv' = \alpha_2 v'$ for $v' \in V/V_2$. Let $\tilde{\mathcal{L}} = f^*(\mathcal{L}_1 \boxtimes \mathcal{L}_2)$, a local system on ${}^0\tilde{G}$. Define $f_i : {}^0\tilde{G} \rightarrow \mathcal{A}'$ ($i = 1, 2$) by $f_1(g, V_2) = \alpha_1 \alpha_2$, $f_2(g, V_2) = \alpha_1$ where α_1, α_2 are as above. Then $\tilde{\mathcal{L}} = f_1^* \mathcal{L}_1 \otimes f_2^* \mathcal{L}$. (We use 5(a).)

We shall assume that \mathcal{L} is *regular* in the following sense: the restriction of \mathcal{E} to

the subgroup $\mathcal{T} = \{1 + \epsilon c; c \in \mathbf{k}\}$ of \mathcal{A}' is not isomorphic to $\bar{\mathbf{Q}}_l$.

Lemma 10. (a) \tilde{G} is an irreducible, smooth variety and $\tilde{G} - {}^0\tilde{G}$ is a smooth irreducible hypersurface in \tilde{G} .

(b) We have $IC(\tilde{G}, \tilde{\mathcal{L}})|_{\tilde{G}-{}^0\tilde{G}} = 0$.

Note that $f_1 : {}^0\tilde{G} \rightarrow \mathcal{A}'$ extends to the whole of \tilde{G} by $f_1(g, V_1, V_2) = \det_{\mathcal{A}}(g : V \rightarrow V)$. Hence $f_1^* \mathcal{L}_1$ extends to a local system on \tilde{G} and we have $IC(\tilde{G}, \tilde{\mathcal{L}}) = f_1^* \mathcal{L}_1 \otimes IC(\tilde{G}, f_2^* \mathcal{E})$. Hence to prove (b) it is enough to show that $IC(\tilde{G}, f_2^* \mathcal{E})$ is zero on $\tilde{G} - {}^0\tilde{G}$.

Let Z (resp. H) be the fibre of the second projection $\tilde{G} \rightarrow X_1$ (resp. $\tilde{G} - {}^0\tilde{G} \rightarrow X_1$) at $V_1 \in X_1$. Since G acts transitively on X_1 it is enough to show that Z is smooth, irreducible, H is a smooth, irreducible hypersurface in Z and $IC(Z, f_2^* \mathcal{E})$ is zero on H (the restriction of f_2 to Z is denoted again by f_2).

Let e_1, e_2 be a basis of V such that $V_1 = \mathbf{k}\epsilon e_1$. The subspaces $V_2 \in X_2$ such that $V_1 \subset V_2$ are exactly the subspaces $V_2^{z', z''} = \mathbf{k}\epsilon e_1 + \mathbf{k}(z' e_1 + z'' \epsilon e_2)$ where $(z', z'') \in \mathbf{k}^2 - \{0\}$. An element $g \in G$ is of the form

$$\begin{aligned} ge_1 &= a_0 e_1 + b_0 e_2 + a_1 \epsilon e_1 + b_1 \epsilon e_2, \\ ge_2 &= c_0 e_1 + d_0 e_2 + c_1 \epsilon e_1 + d_1 \epsilon e_2 \end{aligned}$$

where $a_i, b_i, c_i, d_i \in \mathbf{k}$ satisfy $a_0 d_0 - b_0 c_0 \neq 0$.

The condition that $g\epsilon e_1 \in \mathbf{k}\epsilon e_1$ is $b_0 = 0$. The condition that $gV_2^{z', z''} = V_2^{z', z''}$ is that $z' b_1 + z'' d_0 = a_0 z''$ if $z' \neq 0$ (no condition if $z' = 0$). The condition that the scalars by which g acts on V_1 and $V_2^{z', z''}/V_1$ coincide is $a_0 = d_0$ if $z' = 0$ (no condition if $z' \neq 0$).

We see that we may identify Z with

$$\begin{aligned} \{(a_0, c_0, d_0, a_1, b_1, c_1, d_1; z', z'') \in \mathbf{k}^7 \times (\mathbf{k}^2 - \{0\})/\mathbf{k}^*; \\ a_0 \neq 0, d_0 \neq 0, z' b_1 = z''(a_0 - d_0)\} \end{aligned}$$

and H with the subset defined by $z' = 0$. In this description it is clear that Z is irreducible, smooth and H is a smooth, irreducible hypersurface in Z . The function f_2 takes a point with $z' \neq 0$ to $a_0 + \epsilon(a_1 + z'' z'^{-1} c_0)$. To prove the statement on intersection cohomology we may replace Z by the open subset $z'' \neq 0$ containing H . Thus we may replace Z by

$$Z_1 = \{(a_0, c_0, d_0, a_1, b_1, c_1, d_1; z) \in \mathbf{k}^7 \times \mathbf{k}; a_0 \neq 0, d_0 \neq 0, z b_1 = a_0 - d_0\}$$

and H by the subset defined by $z = 0$. The function f_2 is defined on $Z_1 - H$ by

$$a_0 + \epsilon(a_1 + z^{-1} c_0) = (a_0 + \epsilon a_1)(1 + \epsilon z^{-1} c_0 a_0^{-1}).$$

Thus $f_2 = f_3 f_4$ where f_3 (resp. f_4) is defined on $Z_1 - H$ by $a_0 + \epsilon a_1$ (resp. $1 + \epsilon z^{-1} c_0 a_0^{-1}$). Hence $f_2^* \mathcal{E} = f_3^* \mathcal{E} \otimes f_4^* \mathcal{E}$. Now f_3 extends to Z_1 hence $f_3^* \mathcal{E}$ extends

to a local system on Z_1 . We have $IC(Z_1, f_3^* \mathcal{E} \otimes f_4^* \mathcal{E}) = f_3^* \mathcal{E} \otimes IC(Z_1, f_4^* \mathcal{E})$. It is enough to show that $IC(Z_1, f_4^* \mathcal{E})$ is zero on H . We make the change of variable $c = c_0 a_0^{-1}$. Then Z_1 becomes

$$Z_1 = \{(a_0, c, a_1, b_1, c_1, d_1; z) \in \mathbf{k}^7 \times \mathbf{k}; a_0 \neq 0, a_0 - zb_1 \neq 0\},$$

H is the subset defined by $z = 0$ and $f_4 : Z_1 - H \rightarrow \mathcal{A}'$ is given by $1 + \epsilon z^{-1}c$. Let $\tilde{Z}_1 = \{(a_0, c, a_1, b_1, c_1, d_1; z) \in \mathbf{k}^7 \times \mathbf{k}\}$ and let H_1 be the subset of \tilde{Z}_1 defined by $z = 0$. Then Z_1 is open in \tilde{Z}_1 and f_4 is well defined on $\tilde{Z}_1 - H_1$ by $1 + \epsilon z^{-1}c$. Hence $f_4^* \mathcal{E}$ is well defined on $\tilde{Z}_1 - H_1$. It is enough to show that $IC(\tilde{Z}_1, f_4^* \mathcal{E})$ is zero on H_1 . Let $H' = \{(c, z) \in \mathbf{k}^2; z = 0\}$ and define $f' : \mathbf{k}^2 - H' \rightarrow \mathcal{A}'$ by $f'(c, z) = 1 + \epsilon z^{-1}c$. It is enough to show that $IC(\mathbf{k}^2, f'^* \mathcal{E})$ is zero on H' . Let P be the projective line associate to \mathbf{k}^2 . Then H' defines a point $x_0 \in P$. Since f' is constant on lines, it defines a map $h : P - \{x_0\} \rightarrow \mathcal{A}'$. Since P is 1-dimensional we have $IC(P, h^* \mathcal{E}) = \mathcal{F}$ where \mathcal{F} is a constructible sheaf on P whose restriction to $P - \{x_0\}$ is $h^* \mathcal{E}$. It is enough to show that

(c) the stalk of \mathcal{F} at x_0 is 0;

(d) $H^i(P, \mathcal{F}) = 0$ for $i = 0, 1$.

(Indeed, (c) implies that $IC(\mathbf{k}^2, f'^* \mathcal{E})$ is zero at $(c, 0)$ with $c \neq 0$ and (d) implies that $IC(\mathbf{k}^2, f'^* \mathcal{E})$ is zero at $(0, 0)$.)

Consider the standard \mathbf{F}_q -rational structures on $\mathbf{k}^2, X, \mathcal{A}'$ and let F be the corresponding Frobenius map. We may assume that \mathcal{E} is associated as in §5 to (\mathbf{F}_q, ψ) where $\psi \in \text{Hom}(\mathcal{A}'^F, \bar{\mathbf{Q}}_l^*)$. For any $m \in \mathbf{Z}_{>0}$ there is a unique isomorphism $\phi_m : (F^m)^* \mathcal{E} \xrightarrow{\sim} \mathcal{E}$ which induces the identity on the stalk of \mathcal{E} at 1. The characteristic function of \mathcal{E} with respect to this isomorphism is $a' \mapsto \psi(N_{F^m/F}(a'))$, $a' \in \mathcal{A}'^{F^m}$. Since, by assumption, $\mathcal{E}|_{\mathcal{T}}$ is not isomorphic to $\bar{\mathbf{Q}}_l$, $\psi|_{\mathcal{T}^F}$ is not the trivial character. Hence $\psi \circ N_{F^m/F} : \mathcal{A}'^{F^m} \rightarrow \bar{\mathbf{Q}}_l^*$ is non-trivial on \mathcal{T}^{F^m} . Now ϕ_m induces an isomorphism $\phi'_m : (F^m)^* h^* \mathcal{E} \xrightarrow{\sim} h^* \mathcal{E}$. We show that

(e) $\sum_{x \in P^{F^m} - \{x_0\}} \text{tr}(\phi'_m, (h^* \mathcal{E})_x) = 0$.

An equivalent statement is:

$$\sum_{(c,z) \in \mathbf{F}_{q^m} \times \mathbf{F}_{q^m}^*} (\psi \circ N_{F^m/F})(1 + \epsilon z^{-1}c) = 0,$$

which follows from the fact that $\psi \circ N_{F^m/F} : \mathcal{A}'^{F^m} \rightarrow \bar{\mathbf{Q}}_l^*$ is non-trivial on \mathcal{T}^{F^m} . Introducing (e) in the trace formula for Frobenius, we see that

$$(f) \sum_{i=0}^2 (-1)^i \text{tr}(\phi'_m, H^i(P, \mathcal{F})) = \text{tr}(\phi'_m, \mathcal{F}_{x_0})$$

where \mathcal{F}_{x_0} is the stalk of \mathcal{F} at x_0 and ϕ'_m is in fact equal to $\phi'_1{}^m$ (for $m = 1, 2, 3, \dots$). By Deligne's purity theorem, $H^i(P, \mathcal{F})$ together with ϕ'_1 is pure of weight i ; by Gabber's theorem [BBD], \mathcal{F}_{x_0} together with ϕ'_1 is mixed of weight ≤ 0 . Hence from (f) we deduce that $H^1(P, \mathcal{F}) = 0$, $H^2(P, \mathcal{F}) = 0$ and $\dim H^0(P, \mathcal{F}) = \dim \mathcal{F}_{x_0}$. By the hard Lefschetz theorem [BBD] we have $\dim H^0(P, \mathcal{F}) = \dim H^2(P, \mathcal{F})$. It follows that $H^0(P, \mathcal{F}) = 0$ hence $\mathcal{F}_{x_0} = 0$. This proves (c),(d). The lemma is proved.

Lemma 11. Define $\rho : {}^0\tilde{G} \rightarrow G$ by $(g, V_2) \mapsto g$. Let $K = \rho_! \tilde{\mathcal{L}}$. Let G_0 be the open dense subset of G consisting of all $g \in G$ such that $g : \epsilon V \rightarrow \epsilon V$ is regular,

semisimple. Let $\rho_0 : \rho^{-1}(G_0) \rightarrow G_0$ be the restriction of ρ . Then $\rho_{0!}\tilde{\mathcal{L}}$ is a local system on G_0 . We have $\dim \operatorname{supp} \mathcal{H}^i K < \dim G - i$ for any $i > 0$.

The first assertion of the lemma follows from the fact that ρ_0 is a double covering. To prove the second assertion it is enough to show that, for $i > 0$, the set G_i consisting of the points $g \in G$ such that $\dim \rho^{-1}(g) = i$ and $\oplus_j H_c^j(\rho^{-1}(g), \tilde{\mathcal{L}}) \neq 0$ has codimension $> 2i$ in G .

Consider the fibre $\rho^{-1}(g)$ for $g \in G$. We may assume that, with respect to a suitable \mathcal{A} -basis of V , g can be represented as an upper triangular matrix $\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}$ with a, c in \mathcal{A}' and $b \in \mathcal{A}$. (Otherwise, $\rho^{-1}(g)$ is empty.) There are five cases:

Case 1. $a = d \in \mathcal{A}'$. Then $\rho^{-1}(g)$ consists of two points.

Case 2. $a = d \in \epsilon\mathcal{A}, b \in \mathcal{A}'$. Then $\rho^{-1}(g)$ is an affine line.

Case 3. $a = d \in \epsilon\mathcal{A} - \{0\}, b \in \epsilon\mathcal{A}$. Then $\rho^{-1}(g)$ is a disjoint union of two affine lines.

Case 4. $a = d, b \in \epsilon\mathcal{A} - \{0\}$. Then $\rho^{-1}(g)$ is an affine line.

Case 5. $a = d, b = 0$. Then $\rho^{-1}(g)$ is an affine line bundle over a projective line.

In case 2, we may identify $\rho^{-1}(g), \tilde{\mathcal{L}}|_{\rho^{-1}(g)}$ with $P - \{x_0\}, \mathcal{F}|_{P - \{x_0\}}$ in the proof of Lemma 10. Then the argument in that proof shows that $H_c^j(\rho^{-1}(g), \tilde{\mathcal{L}}) = 0$ for all j . We see that G_1 consists of all g as in case 3 and 4, hence G_1 has codimension 3 in G . We see that G_2 consists of all g as in case 5, hence G_2 has codimension 6 in G . The lemma is proved. Note that without the assumption that \mathcal{L} is regular, the last assertion of the lemma would not hold (there would be a violation coming from g in case 2.)

12. We show:

$$(a) \quad \rho_! \tilde{\mathcal{L}} = IC(G, \rho_{0!} \tilde{\mathcal{L}}).$$

Define $\tilde{\rho} : \tilde{G} \rightarrow G$ by $\tilde{\rho}(g, V_1, V_2) = g$. Clearly, $\tilde{\rho}$ is proper. Let $j : {}^0\tilde{G} \rightarrow G$ be the inclusion. We have $\rho = \tilde{\rho} \circ j$ hence $\rho_! \tilde{\mathcal{L}} = \tilde{\rho}_!(j_! \tilde{\mathcal{L}})$. By Lemma 10, we have $j_! \tilde{\mathcal{L}} = IC(\tilde{G}, \tilde{\mathcal{L}})$ hence $\rho_! \tilde{\mathcal{L}} = \tilde{\rho}_! IC(\tilde{G}, \tilde{\mathcal{L}})$. Since $\tilde{\rho}$ is proper, $\tilde{\rho}_!$ commutes with the Verdier duality \mathfrak{D} . Hence $\mathfrak{D}(\rho_! \tilde{\mathcal{L}}) = \tilde{\rho}_! \mathfrak{D} IC(\tilde{G}, \tilde{\mathcal{L}})$. Hence $\mathfrak{D}(\rho_! \tilde{\mathcal{L}})$ equals $\tilde{\rho}_! IC(\tilde{G}, \tilde{\mathcal{L}}^*)$ up to a shift. Now the same argument that shows $j_! \tilde{\mathcal{L}} = IC(\tilde{G}, \tilde{\mathcal{L}})$ shows also $j_! \tilde{\mathcal{L}}^* = IC(\tilde{G}, \tilde{\mathcal{L}}^*)$. Hence, up to shift, $\mathfrak{D}(\rho_! \tilde{\mathcal{L}})$ equals $\tilde{\rho}_! j_! \tilde{\mathcal{L}}^* = \rho_! \tilde{\mathcal{L}}^*$. Now the argument in Lemma 12 can also be applied to $\tilde{\mathcal{L}}^*$ instead of $\tilde{\mathcal{L}}$ and yields $\dim \operatorname{supp} \mathcal{H}^i \rho_! \tilde{\mathcal{L}}^* < \dim G - i$ for any $i > 0$. Thus, $\rho_! \tilde{\mathcal{L}}$ satisfies the defining properties of $IC(G, \rho_{0!} \tilde{\mathcal{L}})$ hence it is equal to it. This proves (a).

We see that conjecture 8(a) holds for $n = 2, r = 2$.

13. If G is a connected affine algebraic group over \mathbf{k} which is neither reductive nor nilpotent, one cannot expect to have a complete theory character sheaves for G . Assume for example that G is the group of all matrices

$$[a, b] = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

with entries in \mathbf{k} . The group $G(\mathbf{F}_q)$ (for the obvious \mathbf{F}_q -rational structure) has $(q - 1)$ one dimensional representations and one $(q - 1)$ -dimensional irreducible representation. The character of a one dimensional representation can be realized in terms of an intersection cohomology complex (a local system on G), but that of the $(q - 1)$ dimensional irreducible representation appears as a difference of two intersection cohomology complexes, one given by the local system $\bar{\mathbf{Q}}_l$ on the unipotent radical of G and one supported by the unit element of G . A similar phenomenon occurs for G as in §9 and for a $(q^2 - 1)$ -dimensional irreducible representation of $G(\mathbf{F}_q)$.

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